

# Parity Declustering for Fault-Tolerant Storage Systems via $t$ -designs

Son Hoang Dau, Weiya Xi, Chao Jin, Yan Jia, Kheong Sann Chan

**Abstract**—Parity declustering allows faster reconstruction of a disk array when some disk fails. Moreover, it guarantees uniform reconstruction workload on all surviving disks. It has been shown that parity declustering for one-failure tolerant array codes can be obtained via Balanced Incomplete Block Designs. We extend this technique for array codes that can tolerate an arbitrary number of disk failures via  $t$ -designs.

## I. INTRODUCTION

RAID (Redundant Array of Independent Disks) has been widely used as a large-scaled and reliable storage system since its introduction in 1988 [10]. However, the key limitation of the first 6 levels of RAID (RAID-0 to RAID-5) is that system recovery can be possible with only one disk failure. RAID-6 has been proposed as a new RAID standard, which requires that any one or two disk failures can be fixed. Several types of codes that can correct two erasures have been proposed, such as Reed-Solomon (RS) code [19], EVEN-ODD code [3], B-code [24], X-code [25], RDP code [9]. Codes that allow the recovery from more than two failures have also been investigated [11], [12], [14]. The main limitation of RS codes is the high encoding and decoding complexity, which involves computation over finite fields. The other types of codes, called array codes, are preferred by storage system designers due to the fact that their encoding and decoding requires only XOR operations.

The majority of known array codes are MDS (Maximum Distance Separable) codes. MDS array codes have optimal redundancy ( $\delta$  redundant disks are used in a  $\delta$ -erasure-correcting array code). The main issue with them is that when  $\delta$  disks fail, data in every surviving disk has to be read for reconstruction. This results in slow reconstruction time when disk capacities get larger and increases the possibility of another failure, which renders the reconstruction impossible. Moreover, as all

disks must be fully accessed for the recovery purpose, the system operates in its degraded mode: responses to user requests take longer time than usual.

Parity declustering was proposed by Muntz and Lui [17] as a data layout technique that allows faster reconstruction and uniform reconstruction workloads on surviving devices during reconstruction of one disk failure. Here, the reconstruction workload refers to the amount of data that needs to be accessed on the surviving disks in order to reconstruct the data on the failed disk. Faster reconstruction stems from the feature of the data layout that requires only a *partial* access instead of a full access to each surviving disk. In other words, the special layout allows reconstruction of data on a failed disk *without* reading all data in every surviving disk. Muntz and Lui suggested that designing such a layout is a combinatorial block design problem, but gave no further details. Holland and Gibson [13], Ng and Mattson [18] investigated the construction of parity-declustering data layouts from Balanced Incomplete Block Designs (BIBD). The work of Reddy and Banerjee [20] also followed the same approach, even though they focused more on a special type of BIBDs.

For codes that can tolerate  $\delta \geq 2$  disk failures, it is also desirable to have a declustering-parity data layout. More specifically, we want to design a data layout such that when at most  $\delta$  disks fail, only a *portion* of the disk content on each healthy disk needs to be accessed for the recovery process. Moreover, the reconstruction workload is distributed *uniformly* to all surviving disks. There has been several work where parity declustering for  $\delta$ -failure tolerant codes ( $\delta \geq 2$ ) are considered, such as [1] and [2]. However, none of them guarantee the uniform workloads during the reconstruction of more than one disk. Corbett [8] proposed that two (or more) array codes of the same size can be combined into a larger array that has almost uniform reconstruction workloads when one or two disks fail. However, Corbett's method only achieves uniform workloads among the *data* disks, not over all surviving disks (data disks and parity disks). Moreover, his construction produces an array code of a prohibitively large size, which is at least  $\binom{n}{n/2} \times n$ .

We investigate the construction of declustering-parity layouts for codes that tolerate  $t - 1$  disk failures via

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$t$ -designs ( $t \geq 2$ ). In fact, BIBDs, which are used to decluster parities for one-failure tolerant codes, are 2-designs. The main idea is to start with an array code of  $k$  columns that has uniform workloads for reconstruction of every  $s \leq t - 1$  columns. Then, the  $k$  columns of this code are spread out over  $n > k$  disks, using blocks of a  $t$ -design (see Section II for all definitions). As a result, we obtain an array code with  $n$  disks that possesses the following properties. Firstly, in order to recover any  $s \leq t - 1$  disks, only a portion of the disk content, which is a designed parameter, must be read for disk recovery. Secondly, the reconstruction workload is uniformly distributed to every surviving disk. And lastly, the parity units are distributed evenly over all disks, which eliminates hot spots during data update. To the best of our knowledge, this is the first work that extends the well-known parity declustering technique (originally proposed for one-failure tolerant codes) for  $\delta$ -failure tolerant codes, for any  $\delta \geq 1$ .

The paper is organized as follows. Necessary definitions and notations are provided in Section II. In this section, we also review the parity declustering technique for one-failure tolerant codes based on BIBDs. We extend this technique for two-failure tolerant codes via 3-designs in Section III. In Section IV, we discuss the generalization of this idea for codes that can tolerate  $\delta \geq 2$  disk failures. The paper is concluded in Section V.

## II. PRELIMINARIES

Disk arrays spread data across several disks and access them in parallel to increase data transfer rates and I/O rates. Disk arrays are, however, highly vulnerable to disk failures. An array with  $n$  disks is  $n$  times more likely to fail than a single disk [10]. Adding redundancy to a disk array is a natural solution to this problem. *Units* of data on  $k$  disks are grouped together into *parity groups* (or parity stripes). Each parity group consists of  $k - 1$  data units and one parity unit. The parity unit is calculated by taking the XOR-sum of the data units in the same group. The parity unit must be updated whenever a data unit in its group is modified. Therefore, the parity units should be distributed across the array rather than all being located on a small subset of disks. Otherwise we would have the situation where some disks are always busy updating the parity units while the others are totally idle. Ideally, we want to have the same amount of parity units on every disk. This requirement guarantees that the parity update workload is uniformly distributed among all disks. Additionally, it is required that no two units from the same parity group are located on the same disk, so that the disk array can always be recovered from one disk failure.

Disk 0	Disk 1	Disk 2	Disk 3
$D_0$	$D_0$	$D_0$	$P_0$
$D_1$	$D_1$	$P_1$	$D_1$
$D_2$	$P_2$	$D_2$	$D_2$
$P_3$	$D_3$	$D_3$	$D_3$

Fig. 1: An array code with no parity declustering

Let us consider the following example. Suppose there are four disks in the disk array. Each disk is divided into several units. They are either data units ( $D$ ) or parity units ( $P$ ). Each parity group consists of three data units and one parity unit (those that have the same index). The parity unit is equal to the XOR-sum of the data units in the same parity group. The array in Fig. 1 represents the *basic* pattern of the data/parity layout in this disk array. This basic pattern is then repeated many times until every unit in each disk is covered by some pattern. This pattern of data/parity layout is called an *array code* for the disk array. Column  $i$  of the array code corresponds to Disk  $i$  in the disk array that employs the array code. A data/parity entry in Column  $i$  represents a data/parity unit in Disk  $i$ . Without loss of generality, we assume that the disk array consists of only one copy of the data/parity layout from the array code. In other words, we assume that the data/parity layout of the disk array looks completely the same as the data/parity layout of the array code. Then, throughout this work, we often use disks and columns, units and entries, interchangeably.

The array code presented in Fig. 1 can recover *one* missing column. Hence, the disk array that employs this array code can tolerate *one* disk failure. The reconstruction process of the lost column (disk) requires access to *all* entries (units) in every surviving column (disk).

The parity declustering technique for one-failure tolerant array codes based on BIBDs was originally suggested by Muntz and Lui [17] and investigated in details by Holland and Gibson [13], Ng and Mattson [18], and Reddy and Banerjee [20]. Before describing this technique, we need the definitions of  $t$ -designs and BIBDs.

**Definition II.1.** A  $t$ -( $n, k, \lambda$ ) design, a  $t$ -design in short, is a pair  $(\mathcal{X}, \mathcal{B})$  where  $\mathcal{X}$  is a set of  $n$  points and  $\mathcal{B}$  is a collection of  $k$ -subsets of  $\mathcal{X}$  (blocks) with the property that every  $t$ -subset of  $\mathcal{X}$  is contained in exactly  $\lambda$  blocks. A 2-( $n, k, \lambda$ ) design is also called a *balanced incomplete block design* (BIBD).

Given a 2-( $n, k, \lambda$ ) design, we associate disks with

Disk 0	Disk 1	Disk 2	Disk 3	Disk 4
$D_0$	$D_0$	$D_0$	$P_0$	$P_1$
$D_1$	$D_1$	$D_1$	$D_2$	$P_2$
$D_2$	$D_2$	$D_3$	$D_3$	$P_3$
$D_3$	$D_4$	$D_4$	$D_4$	$P_4$

Fig. 2: An array code with parity declustering

points and parity groups with blocks. As an illustrative example, consider a  $2-(5, 4, 3)$  design with  $\mathcal{X} = \{0, 1, 2, 3, 4\}$  and  $\mathcal{B}$  consisting of five blocks:  $\{0, 1, 2, 3\}$ ,  $\{0, 1, 2, 4\}$ ,  $\{0, 1, 3, 4\}$ ,  $\{0, 2, 3, 4\}$ ,  $\{1, 2, 3, 4\}$ . Each block corresponds to one parity group. For instance, the block  $\{1, 2, 3, 4\}$  corresponds a parity group with the (three) data units being located in Disks 1, 2, 3 and the parity unit located in Disk 4. The data layout of the array code is presented in Fig. 2. We can also balance the number of parity units in every column by rotating the array in this figure cyclically five times (see [13]).

Since every two elements in the set  $\{0, 1, 2, 3, 4\}$  appears in precisely three different blocks, every two disks share three pairs of units, where units in each pair belong to the same parity group. Therefore, when one disk fails, precisely three units in each surviving disk need to be read for the recovery of units on the failed disk. Thus, instead of reading 100% units in each surviving disk (as for the array code in Fig. 1), the reconstruction process now reads 75% units in each disk. In other words, by adding one more disk to the array, we can reduce the percentage of data that needs to be read in each surviving disk for recovery. However, we lose the MDS property of the code while spreading out the workload over more disks. Now it requires 1.25 disks worth of parity instead of just one parity disk as in the previous example. Therefore, the parity declustering technique can be considered as a way to sacrifice the efficiency for faster reconstruction time.

The connection between the reconstruction of one-disk failure and a 2-design is elaborated further as follows. If a parity group  $G$  contains a unit from a disk  $D$  then  $D$  is said to be *crossed* by  $G$ . The reconstruction of one unit requires access to all other units in the same parity group. Therefore, in order to have uniform workloads during the reconstruction for one disk failure, every two disks must share the same number of pairs of units that are from the same parity groups. In other words, every two disks must be simultaneously crossed by the same number of parity groups. If disks and parity groups are

associated to points and blocks, respectively, then the aforementioned property of the data layout becomes the familiar requirement for a 2-design: every two points must be simultaneously contained in the same number of blocks. Thus, the parity technique for one-failure tolerant array codes can be summarized as follows:

**Algorithm 1** ([17], [13], [18], [20])

- **Input:**  $n$  is the number of physical disks in the array and  $k$  is the parity group size.
- **Step 1:** Choose a parity group  $G$  with  $k - 1$  data units and one parity unit.
- **Step 2:** Choose a  $2-(n, k, \lambda)$  design  $\mathcal{D} = (\mathcal{X}, \mathcal{B})$ .
- **Step 3:** For each block  $B_i = \{b_{i,0}, \dots, b_{i,k-1}\} \in \mathcal{B}$ ,  $0 \leq i < |\mathcal{B}|$ , create a parity group  $G_i$  as follows. First,  $G_i$  must have the same data-parity pattern as  $G$ . In other words,  $G_i$  has  $k - 1$  data units and one parity unit, and the parity unit is equal to the XOR-sum of the data units. Second, the  $k - 1$  data units of  $G_i$  are located on disks with labels  $b_{i,0}, \dots, b_{i,k-2}$ . The parity unit of  $G_i$  is located on disk with label  $b_{i,k-1}$ .
- **Output:** The  $n$ -disk array with  $|\mathcal{B}|$  parity groups and their layouts according to Step 3.

In the next sections, we generalize this procedure to construct declustered-parity layouts for array codes that tolerate more than one disk failure.

### III. PARITY DECLUSTERING FOR TWO-FAILURE TOLERANT CODES VIA 3-DESIGNS

To extend the parity declustering technique for two-failure tolerant codes, we use balanced 2-parity groups instead of parity groups.

#### A. $\delta$ -Parity Groups

**Definition III.1.** A  $\delta$ -parity group is an MDS  $\delta$ -failure tolerant array code. More formally, a  $\delta$ -parity group is an  $m \times k$  array that satisfies the following conditions:

- (C1) it contains  $(k - \delta)m$  data entries and  $\delta m$  parity entries;
  - (C2) entries in at most  $\delta$  columns can always be reconstructed from the entries in other columns.
- Moreover, if a  $\delta$ -parity group also satisfies the two other conditions
- (C3) for the reconstruction of entries in at most  $\delta$  columns, the number of entries in every other column that contribute to the calculation must always be the same;
  - (C4) the number of parity entries in every column must be the same,

then it is said to be *balanced*. If a  $\delta$ -parity group does not satisfy either (C3) or (C4) then it is said to be

*unbalanced*. We refer to  $k$  as the *size* of the  $\delta$ -parity group.

Note that the condition (C3) depends on the particular reconstruction algorithm used for the  $\delta$ -parity group. Therefore, a  $\delta$ -parity group can be balanced or unbalanced when different reconstruction algorithms are employed. In fact, all MDS two-failure tolerant array codes, such as Reed-Solomon (RS) codes [19], EVENODD [3], RDP [9], B-code [24], P-codes [15], X-codes [25], are 2-parity groups. However, they are not yet balanced in their original form. The *vertical* codes (B-, P-, X-codes), which contain *both* data and parity units in each column, equipped with their conventional reconstruction algorithms for one failure, satisfy (C4) but not (C3). The *horizontal* codes (RS, EVENODD, RDP), which contain *either* data *or* parity units in each column, in their original form satisfy neither (C3) nor (C4). The following example shows how to modify the existing MDS horizontal codes to obtain balanced 2-parity groups.

**Example III.2.** We first consider RDP codes. Let  $p$  be a prime. RDP code for a  $(p + 1)$ -disk array is defined as a  $(p - 1) \times (p + 1)$  array [9] (see Fig 3). Its first  $p - 1$  columns (disks) store data entries (units) and its last two columns (disks) store parity entries (units). The first parity column ( $P$ -column) stores the row-parity entries; each of such entries is equal to the XOR-sum of the data entries on the same row. The second parity column ( $Q$ -column) stores the diagonal-parity entries; each of such entries is equal to the XOR-sum of the data and row-parity entries along some diagonal of the array. Note that one diagonal is not used (called the *missing* diagonal in [9]).

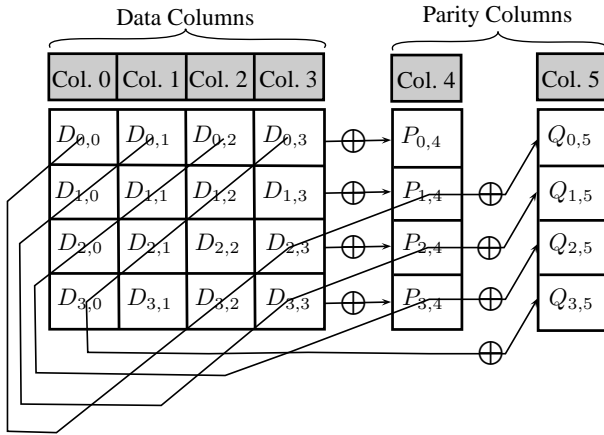


Fig. 3: RDP array with  $p = 5$  (reproduced from [23])

The conventional reconstruction rule for RDP is as

follows. Suppose one column is lost. If it is a data column ( $D$ ), then each of its entries can be recovered by taking the XOR-sum of the data entries in other data columns ( $D$ ) and the row parity entry on the  $P$ -column that belong to the same row. In this way, the  $Q$ -column plays no role in the reconstruction of one lost data column. If the  $P$ -column or the  $Q$ -column is lost, then its entries can be reconstructed by recalculating the parities according to the encoding rule of RDP. Note that the reconstruction of the  $P$ -column does *not* require access to the  $Q$ -column, and vice versa. Hence, the RDP array and its conventional reconstruction rule does not qualify as a balanced 2-parity group. However, we can transform an RDP array into a balanced 2-parity group as follows. Let us first label the data columns by ' $D$ ' and the parity columns by ' $P$ ' and ' $Q$ ', respectively. As an example, the RDP array ( $p = 5$ ) in its simplified form is depicted in Fig. 4.

Col. 0	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
$D$	$D$	$D$	$D$	$P$	$Q$

Fig. 4: An RDP array with  $p = 5$  (simplified layout)

We consider all possible ways to arrange the  $P$ -column and the  $Q$ -column among all  $k$  columns ( $k = p + 1$ ). There are  $k(k - 1)$  such arrangements. If  $k = 6$  then there are  $30 = 6 \times 5$  possible such arrangements. For each of such arrangements of  $P$ - and  $Q$ -columns, we obtain a new array,  $A_i$ ,  $0 \leq i < k(k - 1)$ . We juxtapose all these arrays vertically to obtain a new array  $\mathcal{G}$ , which contains  $k(k - 1)$  times more rows than the original RDP array (see Fig. 5 for the case when  $k = 6$ ).

Col. 0	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
$D$	$D$	$D$	$D$	$P$	$Q$
$D$	$D$	$D$	$D$	$Q$	$P$
$D$	$D$	$D$	$P$	$D$	$Q$
$D$	$D$	$D$	$Q$	$D$	$P$
$D$	$D$	$P$	$D$	$D$	$Q$
$D$	$D$	$Q$	$D$	$D$	$P$
$\vdots$					
$P$	$Q$	$D$	$D$	$D$	$D$
$Q$	$P$	$D$	$D$	$D$	$D$

Fig. 5: A balanced 2-parity group obtained from an RDP array ( $p = 5$ )

Our goal now is to show that the array  $\mathcal{G}$  constructed above, together with RDP's conventional reconstruction rule, in general, is a balanced 2-parity group. The array  $\mathcal{G}$  obviously satisfies (C1), (C2), and (C4). We only need

to verify Condition (C3) for  $\mathcal{G}$ . To recover two missing columns, every other column has to be read in full. Hence, the reconstruction workload for two missing column is already uniform across the columns. To recover one missing column, each of other columns either has to be read in full or is not accessed at all. Therefore, it suffices to regard each column  $D$ ,  $P$ , or  $Q$  as a single entry, or more precisely, a *column-entry*, in  $\mathcal{G}$ , and use the reconstruction rule for RDP as shown in Fig. 6. Those column-entries of  $\mathcal{G}$  correspond to *column-units* on physical disks where each column-unit is actually a column of data/parity units. We actually examine the number of column-entries (instead of entries) on each column of  $\mathcal{G}$  that must be read for column recovery.

We refer to the group of rows in  $\mathcal{G}$  that contains the entries from each array  $\mathcal{G}_i$  as an *extended-row* of  $\mathcal{G}$ . Then  $\mathcal{G}$  has  $k(k-1)$  extended-rows. For instance, in Fig. 5,  $\mathcal{G}$  has 30 extended-rows.

For two distinct columns  $i$  and  $j$  of  $\mathcal{G}$ , we define the following quantities:

- $r_{DQ}$ : the number of extended-rows that has a  $D$  at Column  $i$  and has a  $Q$  at Column  $j$ ;
- $r_{PQ}$ : the number of extended-rows that has a  $P$  at Column  $i$  and has a  $Q$  at Column  $j$ ;
- $r_{QP}$ : the number of extended-rows that has a  $Q$  at Column  $i$  and has a  $P$  at Column  $j$ .

Lost	To be accessed	Not to be accessed
$D$	$D, P$	$Q$
$P$	$D$	$Q$
$Q$	$D$	$P$

Fig. 6: Reconstruction rule for an RDP array

According to the reconstruction rule of RDP arrays (Fig. 6), these extended-rows (that define  $r_{DQ}$ ,  $r_{PQ}$ , and  $r_{QP}$  as above) are *precisely* the extended-rows of  $\mathcal{G}$  on which the recovery of the column-entry in the  $i$ th column does not require access to the column-entry in the  $j$ th column. Therefore, the number of column-entries to be read in column  $j$  during the reconstruction of column  $i$  is precisely

$$k(k-1) - r_{DQ} - r_{PQ} - r_{QP}.$$

Hence, if  $r_{DQ}$ ,  $r_{PQ}$ , and  $r_{QP}$  are all constants for every pair  $(i, j)$  then the reconstruction workload is uniformly distributed to all surviving columns. As the extended-rows of  $\mathcal{G}$  correspond to all possible arrangements of  $P$ -,  $Q$ -, and  $D$ -columns, we have

$$r_{DQ} = k-2, \quad r_{PQ} = 1, \quad r_{QP} = 1,$$

for every pair of columns  $i$  and  $j$  of  $\mathcal{G}$ . Therefore,  $\mathcal{G}$  satisfies (C3).

The same modification also turns an EVENODD array code or a RS code into a balanced 2-parity group. In fact, this method works for every horizontal array code, as long as they have separate parity columns ( $P$ - and  $Q$ -columns) and have reconstruction rules that can be clearly stated in tables similar to the one in Fig. 6.

Note that a simple cyclic rotation does *not* turn a horizontal array code into a balanced 2-parity group. For instance, consider an array obtained by juxtaposing vertically all cyclic rotations of an RDP array with  $p = 5$  as in Fig 7. Suppose the first column is lost. For reconstruction, according to the rule illustrated in Fig. 6, one needs to access *five* column-entries on the second column and only *four* column-entries on the last column. Hence, the reconstruction workload is not distributed uniformly among the surviving columns.

Col. 0	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
$DDDDPQ$	$D$	$D$	$D$	$P$	$Q$
$QDDDDP$	$Q$	$D$	$D$	$D$	$P$
$PQDDDD$	$P$	$Q$	$D$	$D$	$D$
$DPQDDD$	$D$	$P$	$Q$	$D$	$D$
$DDPQDD$	$D$	$D$	$P$	$Q$	$D$
$DDDPQD$	$D$	$D$	$D$	$P$	$Q$

Fig. 7: Rotated RDP array does not form a balanced 2-parity group ( $p = 5$ )

**Definition III.3.** The balanced 2-parity group obtained from an RDP array code as in Example III.2 is called an (balanced) *RDP 2-parity group*. An *EVENODD 2-parity group* and an *RS 2-parity group* are defined in the same way.

**Lemma III.4.** Suppose  $\mathcal{G}$  is a balanced RDP/EVENODD/RS 2-parity group of size  $k$ . Then to reconstruct a missing column of  $\mathcal{G}$ , one needs to read a portion  $\frac{k-2}{k-1}$  of the total content of each other column. In fact, this also holds for every horizontal code that has the same reconstruction rule as the RDP code.

*Proof:* Appendix A. ■

### B. Design of Declustered-Parity Layouts via 3-Designs

Recall that the size  $k$  of a 2-parity group  $\mathcal{G}$  is its number of columns. Each column of  $\mathcal{G}$  corresponds to a column-unit in a physical disk, which is a column of data/parity units.

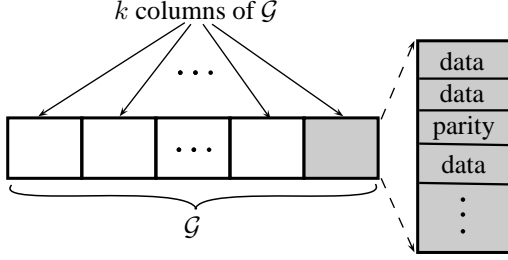


Fig. 8: Simplified layout of a 2-parity group

The following algorithm extends Algorithm 1 to construct declustered-parity layout for two-failure tolerant codes.

#### Algorithm 2

- **Input:**  $n$  is the number of physical disks in the array and  $k$  is the parity group size.
- **Step 1:** Choose a balanced 2-parity group  $\mathcal{G}$  of size  $k$ .
- **Step 2:** Choose a  $3-(n, k, \lambda)$  design  $\mathcal{D} = (\mathcal{X}, \mathcal{B})$ .
- **Step 3:** For each block  $B_i = \{b_{i,0}, \dots, b_{i,k-1}\} \in \mathcal{B}$ ,  $0 \leq i < |\mathcal{B}|$ , create a balanced 2-parity group  $\mathcal{G}_i$  as follows. First,  $\mathcal{G}_i$  must have the same data-parity pattern and the same reconstruction rule as  $\mathcal{G}$ . Second, the  $k$  columns of  $\mathcal{G}_i$  are located on disks with labels  $b_{i,0}, \dots, b_{i,k-1}$ .
- **Output:** The  $n$ -disk array with  $|\mathcal{B}|$  parity groups and their layouts according to Step 3.

Note that even though  $\mathcal{G}_i$ ,  $0 \leq i < |\mathcal{B}|$ , all have the same data-parity pattern of  $\mathcal{G}$ , on the physical disks, they store independent sets of data/parity units. The steps in Algorithm 2 are illustrated in the following example.

**Example III.5.** Suppose  $\mathcal{G}$  is a balanced 2-parity group of size four. For instance,  $\mathcal{G}$  can be obtained from a  $2 \times 4$  RDP array ( $p = 3$ ) using the method described in Example III.2. Then the simplified layout of  $\mathcal{G}$  is as follows (Fig. 9). Each column of  $\mathcal{G}$  actually corresponds to a column of  $24 = 2 \times (4 \times 3)$  parity/data units on a physical disk.

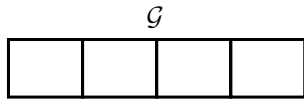


Fig. 9: A balanced 2-parity group of size four

Suppose we have  $n = 8$  physical disks. Consider the following  $3-(8, 4, 1)$  design  $\mathcal{D} = (\mathcal{X}, \mathcal{B})$  where

$$\mathcal{X} = \{0, 1, 2, 3, 4, 5, 6, 7\},$$

and

$$\mathcal{B} = \left\{ \{0, 1, 2, 3\}, \{0, 1, 4, 5\}, \{0, 1, 6, 7\}, \{0, 2, 4, 6\}, \right. \\ \left. \{0, 2, 5, 7\}, \{0, 3, 4, 7\}, \{0, 3, 5, 6\}, \{4, 5, 6, 7\}, \right. \\ \left. \{2, 3, 6, 7\}, \{2, 3, 4, 5\}, \{1, 3, 5, 7\}, \{1, 3, 4, 6\}, \right. \\ \left. \{1, 2, 5, 6\}, \{1, 2, 4, 7\} \right\}.$$

The resulting array code  $\mathcal{C}$  is depicted in Fig. 10. There are 14 2-parity groups in  $\mathcal{C}$ , namely  $\mathcal{G}_i$ ,  $0 \leq i < 14$ . The 2-parity group  $\mathcal{G}_i$  has its columns, labeled by  $i$ , spread across the disks indexed by elements from the block  $B_i \in \mathcal{B}$ ,  $0 \leq i < 14$ . For example, as  $B_{13} = \{1, 2, 4, 7\}$ , the columns of  $\mathcal{G}_{13}$ , labeled by 13, are located on Disk 1, Disk 2, Disk 4, and Disk 7. As each  $\mathcal{G}_i$  is a  $24 \times 4$  array,  $\mathcal{C}$  is actually a  $168 \times 8$  array ( $168 = 7 \times 24$ ).

Disk 0	Disk 1	Disk 2	Disk 3	Disk 4	Disk 5	Disk 6	Disk 7
0	0	0	0	1	1	2	2
1	1	3	5	3	4	3	4
2	2	4	6	5	6	6	5
3	10	8	8	7	7	7	7
4	11	9	9	9	9	8	8
5	12	12	10	11	10	11	10
6	13	13	11	13	12	12	13

Fig. 10: The resulting array code  $\mathcal{C}$

**Theorem III.6.** Algorithm 2 produces an array code that satisfies the following properties

- (P1) it can tolerate at most two simultaneous disk failures;
- (P2) when one or two disks fail, the reconstruction workload is evenly distributed to all surviving disks;
- (P3) every column of  $\mathcal{C}$  has the same amount of parity units.

*Proof:* Appendix B. ■

We now give a high level explanation of how 3-designs and balanced 2-parity groups work well to produce declustered-parity layouts for two-failure tolerant codes.

First, let us examine again the application of 2-designs to one-failure tolerant codes. When one disk fails, it is required that all other disks contribute the same amount of data accesses during the reconstruction process. In

other words, we are examining *pairs of disks* (one failed, one survived) and want to make sure that all of these pairs have the same amount of related data/parity units (Fig. 11). (Related units are units that belong to the same parity group). On the other hand, in a 2-design, a similar-looking condition is applied to *pairs of points*: every pair of points must belong to the same number of blocks. That is how the connection between one-failure tolerant codes and 2-designs could be established.

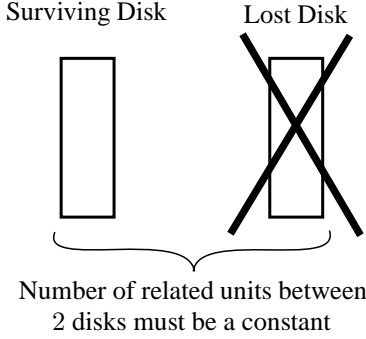


Fig. 11: Requirement for any pair of disks

The problem of designing declustered-parity layouts for two-failure tolerant codes also has a similar requirement. It is required that when one or two disks fail, all surviving disks contribute the same amount of data accesses during the reconstruction process. Suppose two disks fail. We are in fact examining *groups of three disks* (two failed, one survived) and want to make sure that all of these groups have the same amount of “related” data/parity units (Fig. 12). (We use a different meaning here for “related units”. See Appendix B for more details.) If we consider a 3-design, the key property is that every *group of three points* must be contained in the same number of blocks. At first sight, it is not clear how to translate this condition on points/blocks back to the aforementioned condition on disks/groups. However, one can do so with the help from some results in Design Theory. More details can be found in the proof of Theorem III.6 in the Appendix B. Note also that as a 3-design is also a 2-design (see Corollary B.3), uniform workload for reconstruction of one failed disk is automatically guaranteed.

The balance of the 2-parity group used in Algorithm 2 is another key condition to guarantee the balanced reconstruction workload. In the following example, it is demonstrated that Algorithm 2 applied to an unbalanced 2-parity group does *not* produce a code with this property.

**Example III.7.** Suppose the 3-design  $\mathcal{D}$  in Example III.5 and  $\mathcal{G}$ , an RDP  $2 \times 4$  array, are used in Algorithm 2. Note that  $\mathcal{G}$  is an unbalanced 2-parity group with the

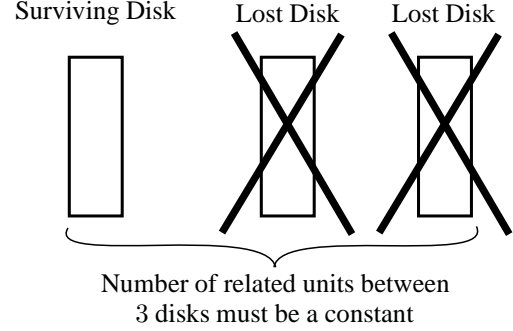


Fig. 12: Requirement for any group of three disks

Disk 0	Disk 1	Disk 2	Disk 3	Disk 4	Disk 5	Disk 6	Disk 7
$D_0$	$D_0$	$P_0$	$Q_0$	$P_1$	$Q_1$	$P_2$	$Q_2$
$D_1$	$D_1$	$D_3$	$D_5$	$P_3$	$P_4$	$Q_3$	$Q_4$
$D_2$	$D_2$	$D_4$	$D_6$	$P_5$	$P_6$	$Q_6$	$Q_5$
$D_3$	$D_{10}$	$D_8$	$D_8$	$D_7$	$D_7$	$P_7$	$Q_7$
$D_4$	$D_{11}$	$D_9$	$D_9$	$P_9$	$Q_9$	$P_8$	$Q_8$
$D_5$	$D_{12}$	$D_{12}$	$D_{10}$	$P_{11}$	$P_{10}$	$Q_{11}$	$Q_{10}$
$D_6$	$D_{13}$	$D_{13}$	$D_{11}$	$P_{13}$	$P_{12}$	$Q_{12}$	$Q_{13}$

Fig. 13: Unbalanced input leads to unbalanced output

reconstruction rule given in Fig. 6. The layout of the resulting code is depicted in Fig. 13.

Suppose Disk 0 and Disk 1 fail. Let us examine the number of column-units on Disk 4 and 6, respectively, that need to be accessed for reconstruction of Disk 0 and Disk 1. According to the reconstruction rule of each group (Fig. 6), *five* column-units on Disk 4 must be accessed, whereas only *one* column-unit on Disk 6 must be accessed (see Fig. 14). Therefore, the workload for reconstruction of the first two disks is not uniformly distributed to the surviving disks.

The reason why Algorithm 2 fails to produce a desired array code in the above example can be explained as follows. Even though the 3-design spreads out the columns of the 2-parity groups evenly among the disks, the columns within each group do not play the same role in the reconstruction of a lost column. More specifically, the  $P$ -column and the corresponding  $Q$ -column do have different roles in the reconstruction of a  $D$ -column. Indeed, according to the reconstruction rule for RDP arrays stated in Fig. 6, the reconstruction of a  $D$ -column

	Disk 0	Disk 1	Disk 4	Disk 6
Group $\mathcal{G}_0$	$D$	$D$	$X$	$X$
Group $\mathcal{G}_1$	$D$	$D$	$\underline{P}$	$X$
Group $\mathcal{G}_2$	$D$	$D$	$X$	$\underline{P}$
Group $\mathcal{G}_3$	$D$	$X$	$\underline{P}$	$Q$
Group $\mathcal{G}_4$	$D$	$X$	$X$	$X$
Group $\mathcal{G}_5$	$D$	$X$	$\underline{P}$	$X$
Group $\mathcal{G}_6$	$D$	$X$	$X$	$Q$
Group $\mathcal{G}_7$	$X$	$X$	$D$	$P$
Group $\mathcal{G}_8$	$X$	$X$	$X$	$P$
Group $\mathcal{G}_9$	$X$	$X$	$P$	$X$
Group $\mathcal{G}_{10}$	$X$	$D$	$X$	$X$
Group $\mathcal{G}_{11}$	$X$	$D$	$\underline{P}$	$Q$
Group $\mathcal{G}_{12}$	$X$	$D$	$X$	$Q$
Group $\mathcal{G}_{13}$	$X$	$D$	$\underline{P}$	$X$

Fig. 14: Related column-units on Disks 0, 1, 4, and 6. The underlined entries are those which must be accessed for reconstruction of Disks 0 and 1. An 'X' in a row labeled by Group  $\mathcal{G}_i$  and in a column labeled by Disk  $j$  means that Disk  $j$  does not contain any column-unit from  $\mathcal{G}_i$ .

requires the access to the  $P$ -column, but not to the  $Q$ -column. For example, even though both Disk 4 and Disk 6 contain column-units from  $\mathcal{G}_3$ , the column-unit  $P_3$  on Disk 4 must be read, while the column-unit  $Q_3$  on Disk 6 is not read (see Fig. 14). If a balanced 2-parity group is used instead, we will not have this problem, as every column in a balanced 2-parity group plays the same role in the reconstruction of a missing column.

### C. Storage Efficiency and Reconstruction Workload Trade-Off

In this subsection we examine the trade-off (of the declustered-parity layout produced by Algorithm 2) between storage efficiency and the workload on every disk during the reconstruction of disk failures. If an  $M \times n$  array code  $\mathcal{C}$  contains  $x$  parity units and  $Mn - x$  data units then we say that the number of *disks worth of parity* in  $\mathcal{C}$  is  $\frac{x}{M}$ . The ratio  $n - \frac{x}{M}$  is called the number of *disks worth of data* of  $\mathcal{C}$ . In other words,  $\mathcal{C}$  uses  $\frac{x}{M}$  disks to store parities and  $n - \frac{x}{M}$  disks to store data.

Another attribute of the array code  $\mathcal{C}$  produced by Algorithm 2 that needs to be examined is the number of rows  $M$ , or *depth*, of  $\mathcal{C}$ . The depth of  $\mathcal{C}$  counts how many units are there in each of its columns. An array with fewer rows results in a smaller-size table being stored in the memory and faster (table) look-up. Furthermore, a code with a smaller depth provides a better local balance (see Schwabe and Sutherland [21]). The depth of  $\mathcal{C}$  depends on  $n$ ,  $k$ , and  $\lambda$ , as shown in the following

theorem. When  $n$  and  $k$  are fixed, the bigger the index  $\lambda$  is, the more rows  $\mathcal{C}$  has. Therefore, 3-designs with smaller  $\lambda$  are preferred.

**Theorem III.8.** *The array code  $\mathcal{C}$  produced by Algorithm 2 satisfies the following properties:*

(P4)  $\mathcal{C}$  has

$$M = m \frac{\lambda(n-1)(n-2)}{(k-1)(k-2)}$$

rows, where  $m$  is the number of rows in the 2-parity group  $\mathcal{G}$ ;

(P5)  $\mathcal{C}$  has  $\frac{(k-2)n}{k}$  disks worth of data and  $\frac{2n}{k}$  disks worth of parity.

Moreover, if an RDP/EVENODD/RS 2-parity group is used in Algorithm 2 then  $\mathcal{C}$  also satisfies the following properties:

- (P6) To reconstruct one failed disk, a portion  $\frac{k-2}{n-1}$  of the total content of each surviving disk needs to be read;
- (P7) To reconstruct two failed disks, a portion  $\frac{(k-2)(2n-k-1)}{(n-1)(n-2)}$  of the total content of each surviving disk needs to be read.

*Proof:* Appendix C. ■

When  $k = n$ , that is, there is no parity declustering involved, Theorem III.8 states the familiar facts about an MDS two-failure tolerant array code:  $\mathcal{C}$  has  $n - 2 = \frac{(n-2)n}{n}$  disks worth of data and  $2 = \frac{2n}{n}$  disks worth of parity; to reconstruct one failed disk, a portion  $\frac{n-2}{n-1}$  of the total content of each surviving disk needs to be read; and to reconstruct two failed disks, each surviving disk needs to be read in full ( $1 = \frac{(n-2)(2n-n-1)}{(n-1)(n-2)}$ ). Note that the second property does not hold for most of known MDS array codes in their original formulations. In fact, it only holds for these codes after some transformation is applied (see Example III.2).

**Example III.9.** In this example, we fix the number of disks in the array to be  $n = 20$ . The parity group size  $k$  varies from 3 to 20. The availability of a particular 3- $(n, k, \lambda)$  design can be found in [7, Part II, Table 4.37]. Note that a  $t$ -design,  $t > 3$ , is also a 3-design. In this table we choose  $\lambda$  to be the smallest possible.

The third and fourth columns show the percentage of data/parity units that have to be read on each surviving disk in order to reconstruct one and two failed disks, respectively. The fifth column presents the number of parity disks to be used when the corresponding parity group size  $k$  is used. The figures in the third, fourth, and fifth columns only depend on  $n$  and  $k$ . As expected, when  $k$  increases, the percentage of units that have to be accessed for disk recovery increases, and the number of parity disks used decreases. Thus, one has to trade



$k$	$\lambda$	1 failure	2 failures	Parity	depth/ $m$
3	1	5.3%	10.5%	13.3	171
4	1	10.5%	20.5%	10.0	57
5	6	15.8%	29.8%	8.0	171
6	10	21.1%	38.6%	6.7	171
7	35	26.3%	46.8%	5.7	399
8	14	31.6%	54.4%	5.0	114
9	28	36.8%	61.4%	4.4	171
10	4	42.1%	67.8%	4.0	19
11	55	47.4%	73.7%	3.6	209
12	55	52.6%	78.9%	3.3	171
13	286	57.9%	83.6%	3.1	741
14	182	63.2%	87.7%	2.9	399
15	273	68.4%	91.2%	2.7	513
16	140	73.7%	94.2%	2.5	228
17	680	78.9%	96.5%	2.4	969
18	136	84.2%	98.2%	2.2	171
19	17	89.5%	99.4%	2.1	19
20	1	94.7%	100%	2.0	1

Fig. 15: Different parity group sizes lead to array codes with different performances ( $n = 20$ )

the storage efficiency for the reconstruction workload (on each disk): increasing storage efficiency, which is good, leads to increasing workload during disk recovery, which is bad, and vice versa. One extreme is when  $k = n$ , where there is no parity declustering. The array code becomes a normal MDS array code, with two disks worth of parities and 100% load on every surviving disk during the reconstruction of two failed disks.

The figures in the last column are the depth of the resulting array code divided by  $m$ , the number of rows of the balanced 2-parity group  $\mathcal{G}$  (see Algorithm 2). These figures depend on  $n$ ,  $k$ , and  $\lambda$ .

The ingredient balanced 2-parity groups  $\mathcal{G}$  of size  $k$  ( $3 \leq k \leq 20$ ) can be constructed using the method presented in Example III.2. This method can be applied to an RS code of length  $k$  for an arbitrary  $k \geq 3$  to obtain a  $(k(k-1)) \times k$  balanced 2-parity group ( $m = k(k-1)$ ). For an EVENODD code [3], this method produces a  $(k(k-1)(k-3)) \times k$  balanced 2-parity group ( $m = k(k-1)(k-3)$ ), for every  $k = p+2$  where  $p$  is a prime. For an RDP code [9], this method produces a  $(k(k-1)(k-2)) \times k$  balanced 2-parity group ( $m = k(k-1)(k-2)$ ), for every  $k = p+1$  where  $p$  is a prime.

**Remark III.10.** Corbett introduced in his patent [8] a method to mix  $n/2$  data disks from one array code with  $n/2$  data disks from another code to produce an array code that has  $n$  data disks. When one or two disks fail,

the reconstruction workload is distributed evenly to all surviving data disks (but not to all data/parity disks). His method actually uses the *complete* 3- $(n, n/2, \lambda)$  design  $(\mathcal{X}, \mathcal{B})$  where all  $(n/2)$ -subsets of  $\mathcal{X}$  are blocks. In fact, any *self-complementary* 3-designs would work well with his construction (a design is self-complementary if it satisfies that  $B \in \mathcal{B}$  if and only if  $\mathcal{X} \setminus B \in \mathcal{B}$ ). The Hadamard 3- $(n, n/2, n/4 - 1)$  design is such a design (see [16]). Using a Hadamard design results in an array code of only  $m(n-1)$  rows, where  $m$  is the depth of the original array codes. By contrast, the construction in [8] produces an array code of an extremely large depth  $m \binom{n}{n/2}$ .

#### IV. PARITY DECLUSTERING FOR

##### $(t-1)$ -FAILURE-TOLERANT CODES VIA $t$ -DESIGNS

The generalization of Algorithm 2 to Algorithm 3 below that works for  $(t-1)$ -failure tolerant codes ( $t \geq 2$ ) is straight-forward.

#### Algorithm 3

- **Input:**  $n$  is the number of physical disks in the array and  $k$  is the parity group size.
- **Step 1:** Choose a balanced  $(t-1)$ -parity group  $\mathcal{G}$  of size  $k$ .
- **Step 2:** Choose a  $t$ -( $n, k, \lambda$ ) design  $\mathcal{D} = (\mathcal{X}, \mathcal{B})$ .
- **Step 3:** For each block  $B_i = \{b_{i,1}, \dots, b_{i,k}\} \in \mathcal{B}$ ,  $0 \leq i < |\mathcal{B}|$ , create a balanced  $(t-1)$ -parity group  $\mathcal{G}_i$  as follows. Firstly,  $\mathcal{G}_i$  must have the same data-parity pattern and the same reconstruction rule as  $\mathcal{G}$ . Secondly, the  $k$  columns of  $\mathcal{G}_i$  are located on disks with labels  $b_{i,1}, \dots, b_{i,k}$ .
- **Output:** The  $n$ -disk array with  $|\mathcal{B}|$  parity groups and their layouts according to Step 3.

Relevant  $t$ -designs can be found in [7, Part II, Table 4.37] and in the references therein. The ingredient balanced  $(t-1)$ -parity group  $\mathcal{G}$  in Algorithm 3 can be constructed by applying the method in Example III.2 to any MDS horizontal array code that tolerates  $t-1$  disk failures. More specifically, suppose that the original array code has  $k-t+1$  data columns ( $D$ ) and  $t-1$  parity columns, namely  $P_i$ -columns,  $i = 1, \dots, t-1$ . There are  $(t-1)! \binom{k}{t-1}$  ways to arrange the parity columns of the original array. For each of such arrangements, we obtain a new array. By juxtaposing vertically all of these  $(t-1)! \binom{k}{t-1}$  arrays, we obtain a balanced  $(t-1)$ -parity group. The proof that the above method works for general  $t$  is almost the same as for  $t = 3$ . For example, for  $t = 4$ , instead of considering just  $r_{DQ}$ ,  $r_{PQ}$ , and  $r_{QP}$ , we now need to consider other quantities, such as  $r_{P_1 P_2}$ ,  $r_{D P_1 P_2}$ , or  $r_{P_1 P_2 P_3}$ . They are, in fact, all constants. Therefore, the arguments go the same way as in Example III.2. We will not provide a detailed proof here.

Except from the well-known RS codes, some other known MDS horizontal  $(t-1)$ -failure tolerant codes ( $t > 3$ ) were studied by Blomer *et al.* [6], Blaum *et al.* [5], [4], Huang and Xu [14].

## V. CONCLUSION

We propose a way to extend the parity declustering technique to multiple-failure tolerant array codes based on balanced  $(t-1)$ -parity groups and  $t$ -designs ( $t \geq 2$ ). Balanced  $(t-1)$ -parity groups can be obtained from any known horizontal array codes that tolerate up to  $t-1$  disk failures. Besides,  $t$ -design is a very well-studied combinatorial object in the theory of Combinatorial Designs. Therefore, one of the advantages of our approach is that we can exploit the rich literature from both Erasure Codes theory and Combinatorial Designs theory.

The second advantage of the approach based on  $t$ -designs is its flexibility. By simply using different  $t$ -designs in the array code construction, one can obtain a variety of different trade-offs between storage efficiency and the recovery time. Note that  $\mathcal{D} = (\mathcal{X}, \mathcal{B})$  where  $\mathcal{B}$  consists of all  $k$ -subset of  $\mathcal{X}$  is a  $t$ -design (called the trivial design) for any  $1 \leq t \leq k \leq n$ . Therefore, for any given number of disks  $n$  and any given parity group size  $k \leq n$ , there always exists a  $t$ -( $n, k, \lambda$ ) design for some  $\lambda$ .

One disadvantage of this approach is that sometimes, the smallest  $t$ -design still has an unacceptably large index  $\lambda$ , which leads to an impractically deep array code. A natural question to ask is whether the depth of the array code, in those cases, can be reduced if we relax some requirements on the array code. A similar question, which is aimed to one-failure tolerant array codes, has already been discussed by Schwabe and Sutherland [21]. Another open question is on the issue of constructing a balanced  $(t-1)$ -parity group. In this work, we show that *horizontal* array codes can be employed to produce such parity groups. However, the question of whether *vertical* array codes can also be useful is still open.

## VI. ACKNOWLEDGMENT

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#### APPENDIX A PROOF OF LEMMA III.4

From Example III.2, for the recovery of one lost column of  $\mathcal{G}$ , one needs to read

$$\begin{aligned} k(k-1) - r_{DQ} - r_{PQ} - r_{QP} \\ = k(k-1) - (k-2) - 1 - 1 \\ = k(k-2) \end{aligned}$$

column-entries in each of the other columns. As each column contains  $k(k-1)$  column-entries, the portion of content of each column that has to be accessed is

$$\frac{k(k-2)}{k(k-1)} = \frac{k-2}{k-1}.$$

#### APPENDIX B PROOF OF THEOREM III.6

##### A. Known Results from Design Theory

The following results from Design Theory are useful in our discussion.

**Theorem B.1.** ([22, Theorem 9.7]) *Suppose that  $(\mathcal{X}, \mathcal{B})$  is a  $t$ -( $n, k, \lambda$ ) design. Suppose that  $Y, Z \subseteq \mathcal{X}$ , where  $Y \cap Z = \emptyset$ ,  $|Y| = i$ ,  $|Z| = j$ , and  $i + j \leq t$ . Then there are exactly*

$$\lambda_i^{(j)} = \frac{\lambda \binom{n-i-j}{k-i}}{\binom{n-t}{k-t}}$$

*blocks in  $\mathcal{B}$  that contain all the points in  $Y$  and none of the points in  $Z$ . In particular,*

$$|\mathcal{B}| = \lambda_0^{(0)} = \frac{\lambda n(n-1)(n-2)}{k(k-1)(k-2)}.$$

**Corollary B.2.** *Suppose that  $(\mathcal{X}, \mathcal{B})$  is a 3-( $n, k, \lambda$ ) design. Then any point  $x$  of  $\mathcal{X}$  is contained in precisely*

$$\lambda_1 = \frac{\lambda(n-1)(n-2)}{(k-1)(k-2)} \quad (1)$$

*blocks.*

*Proof:* Let  $t = 3$ ,  $Y = \{x\}$  and  $Z = \emptyset$  and apply Theorem B.1. ■

**Corollary B.3.** *Suppose that  $(\mathcal{X}, \mathcal{B})$  is a 3-( $n, k, \lambda$ ) design. Then any two distinct points  $x$  and  $y$  in  $\mathcal{X}$  are contained in precisely*

$$\lambda_2 = \frac{\lambda(n-2)}{k-2} \quad (2)$$

*blocks.*

*Proof:* Let  $t = 3$ ,  $Y = \{x, y\}$  and  $Z = \emptyset$  and apply Theorem B.1. ■

**Corollary B.4.** *Suppose that  $(\mathcal{X}, \mathcal{B})$  is a 3-( $n, k, \lambda$ ) design. Suppose that  $x, y$ , and  $z$  are three distinct points in  $\mathcal{X}$ . Then the number of blocks in  $\mathcal{B}$  that contain both  $x$  and  $y$  but not  $z$  is*

$$\lambda_2^{(1)} = \frac{\lambda(n-k)}{k-2}. \quad (3)$$

*Proof:* Let  $t = 3$ ,  $Y = \{x, y\}$ ,  $Z = \{z\}$ , and apply Theorem B.1. ■

Now we are ready to prove Theorem III.6. Let  $\mathcal{C}$  be the array code produced by Algorithm 2. Suppose that in  $\mathcal{G}$  (and hence in every  $\mathcal{G}_i$ ), to recover one (two) missing column, precisely  $\tau_1$  ( $\tau_2$ ) entries have to be read from every other column.

##### B. Proof of $\mathcal{C}$ satisfying (P3)

First note that due to Corollary B.2, each column of  $\mathcal{C}$  contains precisely  $\lambda_1 = \frac{\lambda(n-1)(n-2)}{(k-1)(k-2)}$  column-units. Therefore, each column of  $\mathcal{C}$  contains the same number of units. Also, as each column of  $\mathcal{G}_i$  (that is, each column-unit of  $\mathcal{C}$ ) contains the same number of *parity* units for all  $0 \leq i < |\mathcal{B}|$ , each column of  $\mathcal{C}$  contains the same number of *parity* units. Thus  $\mathcal{C}$  satisfies (P3).

##### C. Proof of $\mathcal{C}$ satisfying (P1)

According to Definition III.1, each 2-parity group can recover up to two missing columns. Moreover, according to Algorithm 2, no two columns of the same group are located (as column-units) in the same column of  $\mathcal{C}$ . Therefore,  $\mathcal{C}$  can tolerate up to two disk failures. Thus  $\mathcal{C}$  satisfies (P1).

##### D. Proof of $\mathcal{C}$ satisfying (P2)

Suppose Disk  $y$  of  $\mathcal{C}$  fails. Let  $x$  be an arbitrary surviving disk of  $\mathcal{C}$ . According to Corollary B.3, in points/blocks language, there are  $\lambda_2$  blocks in  $\mathcal{B}$  that contain both points  $x$  and  $y$ . Translated to disks/groups language, there are  $\lambda_2$  pairs of column-units  $(u_x, u_y)$ , where  $u_x$  is in Disk  $x$ ,  $u_y$  is in Disk  $y$  and  $u_x$  and  $u_y$  are from the same 2-parity group. For such a pair of column-units  $(u_x, u_y)$ , in order to recover  $u_y$ , precisely

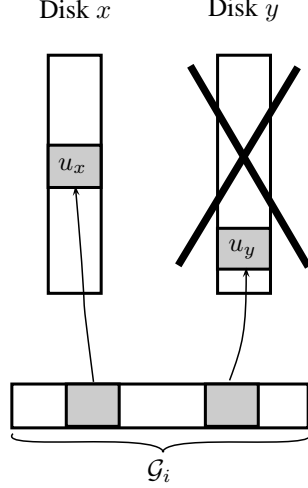


Fig. 16: One disk fails

$\tau_1$  units have to be read from  $u_x$ . Therefore,  $\lambda_2 \tau_1$  units have to be read from Disk  $x$  for the recovery of Disk  $y$ . This number of units is a constant for every pair of Disk  $x$  and  $y$ . Hence, when one disk fails, the reconstruction workload is uniformly distributed to all surviving disks.

Now suppose that Disk  $y$  and Disk  $z$  of  $\mathcal{C}$  fail. Let  $x$  be an arbitrary surviving disk of  $\mathcal{C}$ . A column-unit  $u_x$  in Disk  $x$  is involved in the reconstruction of the two failed disks if and only if one of the following three cases holds.

- **Case 1:** There exist column-units  $u_y$  in Disk  $y$  and  $u_z$  in Disk  $z$  so that  $u_x$ ,  $u_y$ , and  $u_z$  all belong to some 2-parity group  $\mathcal{G}_i$ . In this case, as  $\mathcal{G}_i$  loses two columns, namely  $u_y$  and  $u_z$ ,  $\tau_2$  units have to be read from  $u_x$  for the recovery of the lost columns. According to the definition of a 3-design, there are precisely  $\lambda$  such triple  $(u_x, u_y, u_z)$ .

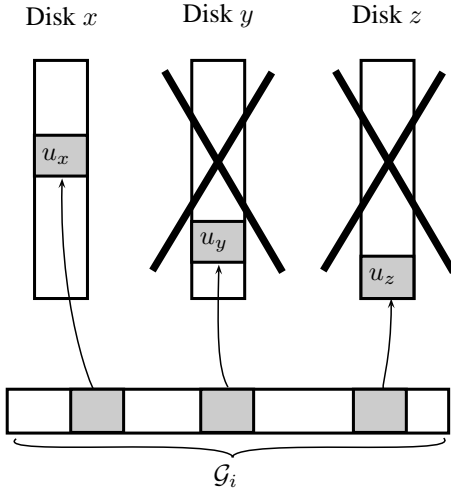


Fig. 17: Case 1

- **Case 2:** There exists a column-unit  $u_y$  in Disk  $y$  such that that  $u_x$  and  $u_y$  belong to some 2-parity group  $\mathcal{G}_i$  and moreover, none of the columns of  $\mathcal{G}_i$  are located in Disk  $z$ . In this case, as  $\mathcal{G}_i$  loses only one column, namely  $u_y$ ,  $\tau_1$  units have to be read from  $u_x$  for the recovery of this lost column. According to Corollary B.4, there are precisely  $\lambda_2^{(1)}$  such pairs  $(u_x, u_y)$ .

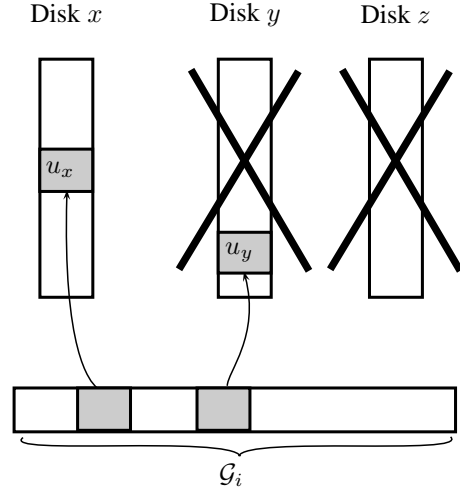


Fig. 18: Case 2

- **Case 3:** There exists a column-unit  $u_z$  in Disk  $z$  such that that  $u_x$  and  $u_z$  belong to some 2-parity group  $\mathcal{G}_i$  and moreover, none of the columns of  $\mathcal{G}_i$  are located in Disk  $y$ . In this case, as  $\mathcal{G}_i$  loses only one column, namely  $u_z$ ,  $\tau_1$  units have to be read from  $u_x$  for the recovery of the lost column. According to Corollary B.4, there are precisely  $\lambda_2^{(1)}$  such pairs  $(u_x, u_z)$ .

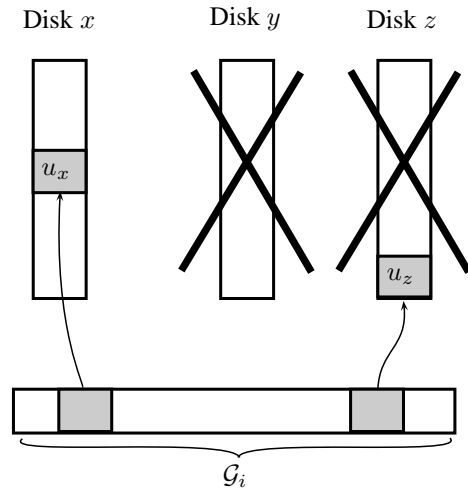


Fig. 19: Case 3

Therefore, in summary, when Disk  $y$  and Disk  $z$  fail, the number of units to be read from Disk  $x$  for the reconstruction is precisely

$$\lambda\tau_2 + 2\lambda_2^{(1)}\tau_1.$$

As this number is a constant for every three distinct disks  $x$ ,  $y$ , and  $z$ , we conclude that when two disks fail, the reconstruction workload is evenly distributed across all surviving disks.

#### APPENDIX C PROOF OF THEOREM III.8

Suppose the 2-parity group  $\mathcal{G}$  employed in Algorithm 2 has  $m$  rows. Recall that  $\tau_i$ ,  $i = 1, 2$ , denotes the number of entries to be read from every other column when  $i$  columns of  $\mathcal{G}$  are lost. If  $\mathcal{G}$  is an RDP/EVENODD/RS 2-parity group then  $\tau_1$  and  $\tau_2$  can be explicitly computed. Indeed, according to Lemma III.4, we have

$$\tau_1 = m \frac{k-2}{k-1}. \quad (4)$$

When two columns of  $\mathcal{G}$  are lost, all  $k-2$  other columns have to be read in full for the recovery of the lost columns. Therefore

$$\tau_2 = m. \quad (5)$$

##### A. Proof of $\mathcal{C}$ satisfying (P4)

According to Corollary B.2, each column of  $\mathcal{C}$  contains precisely  $\lambda_1$  column-entries. Moreover, each of these column-entries consists of  $m$  entries. Therefore, each column of  $\mathcal{C}$  consists of

$$M = m\lambda_1 = m \frac{\lambda(n-1)(n-2)}{(k-1)(k-2)}$$

entries.

##### B. Proof of $\mathcal{C}$ satisfying (P5)

We need to show that  $\mathcal{C}$  has  $\frac{(k-2)n}{k}$  disks worth of data and  $\frac{2n}{k}$  disks worth of parity.

There are  $|\mathcal{B}|$  2-parity balanced groups and each group consists of  $2m$  parity units (see Definition III.1). Therefore, the total number of parity units in  $\mathcal{C}$  is  $2m|\mathcal{B}|$ . Therefore,  $\mathcal{C}$  contains

$$\frac{2m|\mathcal{B}|}{M} = \frac{2|\mathcal{B}|}{\lambda_1} = \frac{2 \frac{\lambda n(n-1)(n-2)}{k(k-1)(k-2)}}{\frac{\lambda(n-1)(n-2)}{(k-1)(k-2)}} = \frac{2n}{k}$$

disks worth of parity. We deduce that  $\mathcal{C}$  contains

$$n - \frac{2n}{k} = \frac{(k-2)n}{k}.$$

disks worth of data.

##### C. Proof of $\mathcal{C}$ satisfying (P6)

We need to prove that if  $\mathcal{G}$  is an RDP/EVENODD/RS 2-parity group then in order to reconstruct one failed disk, a portion  $\frac{k-2}{n-1}$  of the total content of each surviving disk needs to be read.

Suppose one column of  $\mathcal{C}$  is lost. According to Appendix B,  $\lambda_2\tau_1$  entries must be read from each other column for the reconstruction of the missing column. Since each column of  $\mathcal{C}$  consists of  $M$  entries, a portion

$$\frac{\lambda_2\tau_1}{M} = \frac{\frac{\lambda(n-2)}{k-2}m \frac{k-2}{k-1}}{\frac{\lambda(n-1)(n-2)}{(k-1)(k-2)}m} = \frac{k-2}{n-1}$$

of the total content of each surviving disk must be read.

##### D. Proof of $\mathcal{C}$ satisfying (P7)

We need to show that if  $\mathcal{G}$  is an RDP/EVENODD/RS 2-parity group then in order to reconstruct two failed disks, a portion  $\frac{(k-2)(2n-k-1)}{(n-1)(n-2)}$  of the total content of each surviving disk needs to be read.

Suppose two columns of  $\mathcal{C}$  are lost. According to Appendix B,  $\lambda\tau_2 + 2\lambda_2^{(1)}\tau_1$  entries must be read from each other column for the reconstruction of the two missing columns. Thus, a portion

$$\frac{\lambda\tau_2 + 2\lambda_2^{(1)}\tau_1}{M} \quad (6)$$

of the total content of each surviving column needs to be read for the recovery of two columns of  $\mathcal{C}$ . Substituting (4), (5), (1), and (3) into (6), the ratio in this equation can be simplified to

$$\frac{(k-2)(2n-k-1)}{(n-1)(n-2)}.$$